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On stimulated electromagnetic shock radiation (SESR)

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Abstract. We have studied the mechanism of stimulated electromagnetic shock radiation (SESR) caused by the interaction between a relativistic charged particle (electron) and an externally applied electromagnetic plane wave in a dielectric under superphase conditions. We have introduced a relativistic generalisation and considered the time-dependent position of the electron in the field of the incident electromagnetic wave. We have performed classical relativistically invariant calculations for electromagnetic fields of Cherenkov radiation (CR) and SESR in the linear approximation using the method of Fourier transforms. The expressions derived are more general and are consistent with earlier results in special situations. In the linear approximation, the SESR term gets modified and an additional longitudinal SESR term is obtained. There is no change in the transverse SESR term.

1. Introduction

There has been considerable interest in recent years in the study of stimulated radiation resulting from the interaction of relativistic electron beams with coherent electromagnetic fields in polarisable media. In a series of interesting papers [1-5] Schneider and Spitzer have proposed and discussed the basic characteristics of the mechanism of SESR. It is produced by the interaction, in a polarisable medium and under supercritical conditions, of relativistic electrons with counterflowing coherent electromagnetic waves. It involves a synergism between two known phenomena, the Doppler shift in Compton backscattering from relativistic electrons in a vacuum and the formation of a shock by a material body moving in a medium at a speed greater than that of the waves it produces in the medium. It results in the generation of intense radiation in the form of a shock front, with frequencies shifted markedly from that of the incident wave. Schneider and Spitzer [5] claim that with SESR it may be possible to generate a more efficient continuously tunable source of very high frequency (beyond the UV and into the x-ray regime) coherent radiation having a very high degree of monochromaticity. Using different methods Soln [6] and Zachery [7, 8] have also studied SESR. However, experimental verification of the SESR effect has not yet been reported. Even then the process needs to be studied with more exact calculations, because of its possible use for generating coherent upshifted electromagnetic radiation in the frequency region not covered by existing sources.

2. General outline

In this paper, using the method of Fourier transforms, we have obtained leading-order solutions (linear response of the electron to the incident wave) of Maxwell's equations

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which specify the electromagnetic fields that generate CR and SESR in the simplest configuration, namely that of a plane monochromatic wave (of amplitudes $\bar{E}_0$, $\bar{B}_0$ and frequency ω_0) colliding head on with an electron with velocity $\bar{u}$ (parallel to the $\hat{z}$ axis) travelling under supercritical conditions ($u > c/n$, $n = (\varepsilon\mu)^{1/2}$ = refractive index) in a non-magnetic homogeneous isotropic dispersionless dielectric medium of infinite extent. We have used standard notation and Gaussian units throughout this paper.

Following Schneider and Spitzer [5], using Fourier transforms of all quantities, namely $\bar{E}$, $\bar{B}$, $\bar{j}$ and ρ appearing in Maxwell's equations, their general solutions can be written as

$$\bar{E}(\bar{X}, t) = \frac{4\pi i}{(2\pi)^4} \int d^3K\, d\omega \exp[i(\omega t - \bar{K}\cdot\bar{X})] \frac{[\omega \bar{j}(\bar{K}, \omega) - c^2\bar{K}\rho(\bar{K}, \omega)/\varepsilon(\bar{K}, \omega)]}{\omega^2\varepsilon(\omega) - c^2K^2} \tag{1}$$

$$\bar{B}(\bar{X}, t) = \frac{-4\pi c}{(2\pi)^4} \nabla X \int d^3K\, d\omega \exp[i(\omega t - \bar{K}\cdot\bar{X})] \frac{\bar{j}(\bar{K}, \omega)}{\omega^2\varepsilon(\omega) - c^2K^2}. \tag{2}$$

These expressions are general insofar as external sources ($\bar{j}$ and ρ) are concerned, i.e. they hold for an arbitrary distribution of electrons in the incident beam.

In the situation under consideration we can write the charge and current density as

$$\rho(\bar{X}, t) = -e\delta(\bar{X} - \bar{R}(t)) \tag{3a}$$

$$\bar{j}(\bar{X}, t) = -e\bar{V}(t)\delta(\bar{X} - \bar{R}(t)) \tag{3b}$$

where e is the electronic charge, $\bar{v}(t) = d\bar{R}/dt$ and $\bar{R}(t)$ is the position of the incident electron in the field of the incident electromagnetic wave.

The Lorentz force on the incident electron due to the incident electromagnetic wave is

$$\frac{d}{dt}(\gamma m_0 \bar{V}) = -e[\bar{E}_i(\bar{R}, t) + (\bar{V}/c)X\bar{B}_i(\bar{R}, t)] \tag{4}$$

where m_0 is the electron rest mass, $\gamma = (1 - V^2/c^2)^{-1/2}$,

$$\bar{E}_i = \bar{E}_0 \sin(\omega_0 t - \bar{K}_0 \cdot \bar{R})$$

and

$$\bar{B}_i = \bar{B}_0 \sin(\omega_0 t - \bar{K}_0 \cdot \bar{R})$$

are, respectively, the electric and magnetic fields of the incident wave at the electron's position:

$$\bar{B}_i = n\hat{K}_0 X \bar{E}_i \qquad \bar{K}_0 = -(\omega_0 n/c)\hat{z}.$$

We neglect magnetic field effects and the electron's energy changes, but include everywhere the electron's position changes (which was not done earlier in [5]) due to the incident electromagnetic wave. For the electron's velocity and position in the presence of a counterflowing electromagnetic wave we obtain the following equations (see appendix 1) which remain valid even when the electron's motion is relativistic [9, equation (7)]†

$$\bar{V}(t) = (0, +(v_u/\gamma^2)\cos\Omega t, u) \tag{5}$$

$$\bar{R}(t) = (0, +(v_u/\gamma^2\Omega)\sin\Omega t, ut) \tag{6}$$

† In (A8) of [5], addition of two velocities is non-relativistic and at the same time the factor, namely $(1 - 1/\varepsilon\beta^2)$, has been taken out of the integral that appeared in (A19.9) because the authors have approximated the factor, namely uK_z/ω, by one which is valid only for ultrarelativistic particles. We have removed this discrepancy here.

where

$$v_u = \frac{eE_0}{m_0\Omega} \qquad \Omega = \omega_0(1+\beta n) \qquad \beta = |\bar{V}/c| \sim |\bar{u}/c|.$$

Substituting equations (5) and (6) in (3a) and (3b), evaluating Fourier transforms of charge and current densities using properties of δ functions and the relation for the Bessel function, namely

$$\exp(\pm \mathrm{i}a \sin\theta) = \sum_{l=-\infty}^{+\infty} J_l(a)\exp(\pm \mathrm{i}la)$$

where l is an integer, we obtain the Fourier transform of charge density as

$$\rho(\bar{K}, \omega) = -2\pi e \sum_{l=-\infty}^{+\infty} \delta(\omega - K_z u - l\Omega) J_l(K_y y_0) \tag{7}$$

and the Fourier transform of current density as

$$\begin{aligned}\bar{j}(\bar{K}, \omega) = & -2\pi e u\hat{z} \sum_{l=-\infty}^{+\infty} \delta(\omega - K_z u - l\Omega) J_l(K_y y_0) \\ & - \hat{y} e\pi\Omega y_0 \sum_{l=-\infty}^{+\infty} J_l(K_y y_0)\{\delta[\omega - K_z u - (l+1)\Omega] + \delta[\omega - K_z u - (l-1)\Omega]\} \\ = & \hat{z} j_{\mathrm{L}}(\bar{K}, \omega) + \hat{y} j_{\mathrm{T}}(\bar{K}, \omega)\end{aligned} \tag{8}$$

where $y_0 = v_u/\gamma^2\Omega$, and the suffices L and T respectively denote the 'longitudinal' and 'transverse' components with respect to the electron's velocity direction.

Hence we can derive the desired electromagnetic fields by substituting the source densities given by the above equations in equations (1) and (2) and evaluating the multiple integrals appearing therein under supercritical conditions and retaining only terms linear in v_u. These fields may further be used to calculate the energy radiated through CR and SESR.

3. Calculations

Here we present calculations for the electric field. Substituting equations (7) and (8) in equation (1), we obtain

$$\bar{E}(\bar{X}, t) = \hat{z} E_{\mathrm{L}}(\bar{X}, t) + \hat{y} E_{\mathrm{T}}(\bar{X}, t) \tag{9}$$

where

$$E_{\mathrm{L}}(\bar{X}, t) = \frac{2e\beta}{(2\pi)^2 c} \frac{\partial}{\partial t} I_{\mathrm{L}}(\Omega) \tag{10a}$$

where

$$\begin{aligned} I_{\mathrm{L}}(\Omega) = \sum_{l=-\infty}^{+\infty} \int & \mathrm{d}^3K \, \mathrm{d}\omega \exp[\mathrm{i}(\omega t - \bar{K}\cdot\bar{X})] J_l(K_y y_0) \\ & \times \left(1 - \frac{1}{\varepsilon\beta^2}\frac{uK_z}{\omega}\right) \frac{\delta(\omega - K_z u - l\Omega)}{K^2 - \varepsilon\omega^2/c^2}\end{aligned} \tag{10b}$$

and

$$E_T(\bar{X}, t) = \frac{e y_0 \Omega}{(2\pi)^2 c^2} \frac{\partial}{\partial t} I_T(\Omega) \tag{11a}$$

where

$$I_T(\Omega) = \sum_{l=-\infty}^{+\infty} \int d^3K \, d\omega \, \exp[i(\omega t - \bar{K} \cdot \bar{X})] J_l(K_y y_0) \times \left(\frac{\delta(\omega - K_z u - (l+1)\Omega) + \delta(\omega - K_z u - (l-1)\Omega)}{K^2 - \omega^2 \varepsilon / c^2} \right). \tag{11b}$$

The evaluation of the integrals in equations (10b) and (11b) is rather complicated. We give the major steps and omit the details of the calculations.

We assume $\mu = 1$, $\varepsilon =$ constant and $\varepsilon\beta^2 > 1$ throughout. We also note that because of the electron's motion under supercritical condition the region of integration is restricted to inside the CR cone. Introducing cylindrical coordinates, namely $\bar{K}(K_\rho, \phi, K_z)$ and $\bar{X}(\rho, \phi', z)$, equation (10b) takes the following form:

$$I_L(\Omega) = \sum_{l=-\infty}^{+\infty} \int_0^\infty dK_\rho \, K_\rho \int_{-\infty}^{+\infty} dK_z \int_0^{2\pi} d\phi \times \int_{-\infty}^{+\infty} d\omega \frac{\exp(i\omega t)\delta(\omega - K_z u - l\Omega)}{K_z^2 + K_\rho^2 - \varepsilon\omega^2/c^2} \left(1 - \frac{1}{\varepsilon\beta^2} \frac{uK_z}{\omega}\right) \times \exp[-iK_z z - iK_\rho \rho \cos(\phi - \phi')] J_l(K_\rho y_0 \sin\phi). \tag{12}$$

The integral WRT ω appearing in the above equation is calculated using properties of the δ function and the integral WRT ϕ is calculated using the following identities [10]:

$$m \sin(\omega t + \phi) + n \cos(\omega t + \phi) \equiv (m^2 + n^2)^{1/2} \sin(\omega t + \phi + \partial)$$

$$m \sin(\omega t + \phi) - n \cos(\omega t + \phi) \equiv (m^2 + n^2)^{1/2} \sin(\omega t + \phi - \partial)$$

where $\partial = \tan^{-1}(n/m)$ and the integral [11]

$$\int_0^{\pi/2} J_\mu(\nu z \sin t) \cos(\nu x \cos t) \, dt = \frac{\pi}{2} J_{\mu/2}\left(\nu \frac{(x^2 + z^2)^{1/2} + x}{2}\right) J_{\mu/2}\left(\nu \frac{(x^2 + z^2)^{1/2} - x}{2}\right)$$

$$\mathrm{Re}\, \mu > -1 \qquad \mathrm{Re}\, z > 0.$$

Putting $\phi' = 0$, for simplicity, we obtain for the ϕ integral (see appendix 2)

$$\int_0^{2\pi} d\phi \exp[-iK_\rho \rho \cos(\phi - \phi')] J_l(y_0 K_\rho \sin\phi) = 2\pi[J_0(EK_\rho) J_0(FK_\rho) + 2J_{l/2}(EK_\rho) J_{l/2}(FK_\rho)] \tag{13}$$

where

$$E = \tfrac{1}{2}[\rho + (y_0^2 + \rho^2)^{1/2}] \qquad F = \tfrac{1}{2}[(y_0^2 + \rho^2)^{1/2} - \rho]$$

$$l = +2, +4, +6, \ldots.$$

In this way, after evaluating the integrals WRT ω and ϕ equation (12) takes the following form:

$$I_L(\Omega) = 2\pi \sum_{l=2,4}^{\infty} \exp(+\mathrm{i}l\Omega t)\left[\int_0^{\infty} K_\rho \,\mathrm{d}K_\rho\, G(K_\rho)\left(I_1(K_z) - \frac{u}{\varepsilon\beta^2} I_2(K_z)\right)\right] \tag{14}$$

where

$$G(K_\rho) = J_0(EK_\rho)J_0(FK_\rho) + 2J_{l/2}(EK_\rho)J_{l/2}(FK_\rho)$$

$$I_1(K_z) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}K_z \exp(\mathrm{i}u\tau K_z)}{aK_z^2 + bK_z + d} \qquad I_2(K_z) = \int_{-\infty}^{+\infty} \frac{K_z\,\mathrm{d}K_z \exp(\mathrm{i}u\tau K_z)}{(K_z u - l\Omega)(aK_z^2 + bK_z + d)}$$

$$a = 1 - \varepsilon\beta^2 < 0 \qquad b = 2\varepsilon l\Omega u/c^2 \qquad d = K_\rho^2 - \varepsilon l^2\Omega^2/c^2 \qquad \tau = t - z/u > 0.$$

Next we evaluate the integral WRT K_z by contour integration [12] and obtain

$$I_1(K_z) = 2\pi\gamma_s \frac{\exp(-\mathrm{i}Au\tau)}{(K_\rho^2 + B)^{1/2}} \sin[u'(K_\rho^2 + B)^{1/2}] \tag{15}$$

$$I_2(K_z) = \frac{2\pi\mathrm{i}}{ua}\left(\frac{K_1 \exp(\mathrm{i}K_1 u\tau)}{(K_1 - K_2)(K_1 - K_3)} + \frac{K_2 \exp(\mathrm{i}K_2 u\tau)}{(K_2 - K_1)(K_2 - K_3)} + \frac{K_3 \exp(\mathrm{i}K_3 u\tau)}{(K_3 - K_1)(K_3 - K_2)}\right) \tag{16}$$

where

$$K_1 = -A - \gamma_s(K_\rho^2 + B)^{1/2} \qquad K_2 = -A + \gamma_s(K_\rho^2 + B)^{1/2} \qquad K_3 = -l\Omega/u$$

$$A = \varepsilon l\Omega u\gamma_s^2/c^2 \qquad B = \varepsilon l^2\Omega^2\gamma_s^2/c^2 \qquad u' = \gamma_s u\tau$$

$$\gamma_s = (\varepsilon\beta^2 - 1)^{-1/2} = (-a)^{-1/2}.$$

Lastly, to evaluate the integral WRT K_ρ we make use of the following relation for Bessel functions [13]:

$$J_\mu(RZ)J_\nu(\gamma Z) = \frac{R^\mu \gamma^\nu}{\pi}\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \exp[\mathrm{i}\theta(\mu - \nu)]\left(\frac{2\cos\theta}{R^2 \mathrm{e}^{\mathrm{i}\theta} \pm \gamma^2 \mathrm{e}^{-\mathrm{i}\theta}}\right)^{(\mu+\nu)/2}$$
$$\times J_{\mu+\nu}\{Z[2\cos\theta(R^2 \mathrm{e}^{\mathrm{i}\theta} + \gamma^2 \mathrm{e}^{-\mathrm{i}\theta})]^{1/2}\} \tag{17}$$

where $\mathrm{Re}(\mu + \nu) > -1$ and R, γ are real and positive. Using (17) we notice that the second term in $G(K_\rho)$, namely $J_{l/2}(EK_\rho)J_{l/2}(FK_\rho)$ can be omitted, because

$$J_{l/2}(EK_\rho)J_{l/2}(FK_\rho) \propto E^{l/2}F^{l/2} \qquad l = +2, +4, \ldots$$

and $E \sim \rho$, $F \sim 0$ when first-order terms in v_u are kept. Therefore, in the linear approximation only $l = 0$ terms will contribute to the final result.

Further, we make use of the following relations [14]:

$$\int_0^{\infty} \sin(xy)J_\nu(ax)J_\nu(bx)\,\mathrm{d}x = 0 \qquad \text{if } 0 < y < b - a$$

$$= \frac{1}{2(ab)^{1/2}} P_{\nu-1/2}(A) \qquad \text{if } b - a < y < b + a$$

$$= \frac{-1}{\pi(ab)^{1/2}} \cos(\nu\pi) Q_{\nu-1/2}(-A) \qquad \text{if } b + a < y < \infty \tag{18}$$

where $A=(b^2+a^2-y^2)/2ab$, P_ν and Q_ν are Legendre functions of the first and second kind, respectively,

$$\int_0^\infty \frac{x\,\mathrm{d}x}{\beta^2+x^2} J_\nu(xy)J_\nu(ax) = I_\nu(y\beta)K_\nu(a\beta) \qquad \text{if } 0<y<a$$

$$= I_\nu(a\beta)K_\nu(y\beta) \qquad \text{if } a<y<\infty \tag{19}$$

and

$$\mathrm{Re}\,\beta>0 \qquad a>0 \qquad \mathrm{Re}\,\nu>-1$$

where I_ν and K_ν are modified Bessel functions of the first and second kind, respectively, and

$$\int_0^\infty J_0[b(x^2-a^2)^{1/2}]\sin(xy)\,\mathrm{d}x = 0 \qquad \text{if } 0<x<a,\ 0<y<b,\ b>0$$

$$=(y^2-b^2)^{-1/2}\cos[a(y^2-b^2)^{1/2}] \qquad \text{if } a<x<\infty,\ b<y<\infty,\ b>0. \tag{20}$$

To complete all integrations the additional integral WRT θ, which enters into the expression because of the use of equation (17), can be worked out using contour integration to give

$$\int_{-\pi/2}^{+\pi/2} \mathrm{d}\theta \frac{\cos[\sqrt{B}(u'^2-A_\theta^2)^{1/2}]}{(u'^2-A_\theta^2)^{1/2}} = \frac{\cos\sqrt{B}\,q}{q} \tag{21}$$

where

$$A_\theta = [2\cos\theta(E^2\,\mathrm{e}^{\mathrm{i}\theta}+F^2\,\mathrm{e}^{-\mathrm{i}\theta})]^{1/2} \sim (2\cos\theta\,\mathrm{e}^{\mathrm{i}\theta}\rho^2)^{1/2}$$

$$q=(u'^2-\rho^2)^{1/2}>0.$$

Thus using equations (15)-(21) for evaluating integrals WRT K_ρ that appeared in equation (14), we obtain

$$I_{\mathrm{L}}(\Omega) = \frac{4\pi^2\gamma_s}{y_0} P_{-1/2}\left(1-\frac{2q^2}{y_0^2}\right)^{1/2} - \frac{4\pi^2\gamma_s}{\varepsilon\beta^2}\frac{\cos(\sqrt{B}\,q)}{q} \tag{22}$$

where $q>0$, $\tau>0$ ($q=0$ on the shock front, i.e. CR cone).

Simplifying the first term in (22) using the following properties of special functions [13, 15]

$$P_\nu(-Z) = \exp(\pm\nu\pi\mathrm{i})P_\nu(Z) - \frac{2}{\pi}\sin\nu\pi Q_\nu(Z)$$

$(+\nu\pi\mathrm{i}$ with $Z<0$, $-\nu\pi\mathrm{i}$ with $Z>0)$

$$P_{-1/2}(Z) = \frac{2}{\pi}\left(\frac{2}{Z+1}\right)^{1/2} K\left[\left(\frac{Z-1}{Z+1}\right)^{1/2}\right]$$

$$Q_{-1/2}(Z) = \left(\frac{2}{Z+1}\right)^{1/2} K\left[\left(\frac{2}{Z+1}\right)^{1/2}\right]$$

where $K(Z)$ denotes the elliptic integral, and

$$K(m) = \frac{\pi}{2}\left[1+(\tfrac{1}{2})^2 m + \left(\frac{1\times 3}{2\times 4}\right)^2 m^2 + \ldots\right] \qquad |m|<1$$

and taking $B \sim 0$ and substituting equation (22) in equation (10a) we obtain the longitudinal component of the electric field as

$$E_L(\bar{X}, t) = \frac{2e\beta\gamma_s}{c}\left(1 - \frac{1}{\varepsilon\beta^2}\right)\frac{\partial}{\partial t}\left(\frac{1}{q}\right) + \frac{e\beta\gamma_s y_0}{c}\frac{\partial}{\partial t}\left(\frac{1}{q^2}\right) \tag{23}$$

where $q > 0$, $\tau > 0$.

Proceeding on similar lines (as described above to integrate $I_L(\Omega)$) we have calculated the multiple integrals that appear in equation (11b) and obtained, after some simplification,

$$I_T(\Omega) = 8\pi^2\gamma_s \cos(\Omega_s t - Kz)\frac{\cos(\alpha q)}{q} \tag{24}$$

where

$$\alpha = \frac{\sqrt{\varepsilon}\,\mu\gamma_s}{c} \qquad K = \Omega(\gamma_s^2 + 1)/u \qquad \Omega_s = \gamma_s^2\Omega \qquad q > 0, \tau > 0.$$

Substituting equation (24) in equation (11a) we obtain the transverse component of the electric field

$$E_T(\bar{X}, t) = \frac{2ey_0\Omega}{c^2}\frac{\partial}{\partial t}\left(\cos(\Omega_s t - Kz)\frac{\cos(\alpha q)}{q}\right) \tag{25}$$

where $q > 0$, $\tau > 0$.

Thus using equations (23) and (25) in (9) we finally obtain

$$\bar{E}(\bar{X}, t) = \hat{z}\frac{2e\beta\gamma_s}{c}\left(1 - \frac{1}{\varepsilon\beta^2}\right)\frac{\partial}{\partial t}\left(\frac{1}{q}\right) + \hat{z}\frac{e\beta\gamma_s y_0}{c}\frac{\partial}{\partial t}\left(\frac{1}{q^2}\right) + \hat{y}\frac{2ey_0\Omega}{c^2}\frac{\partial}{\partial t}\left(\cos(\Omega_s t - Kz)\frac{\cos(\alpha q)}{q}\right) \tag{26}$$

where $q > 0$, $\tau > 0$.

4. Analysis

We note that, for evaluating the time derivative, we have to insert explicitly the physical conditions, namely there is no field ahead of the particle ($\tau > 0$), and the fields are confined to the region inside the CR cone ($q > 0$) in expression (26) with the help of the step function, namely

$$\theta(x) = 0 \qquad \text{if } x < 0$$
$$= 1 \qquad \text{if } x > 0.$$

Here we are not going into the calculations for radiated power, so we do not express our result in the above way.

The first term of our result specified by equation (26) is identified as the CR term, because it matches with the expression for the longitudinal electric field derived by Tamm [16]. The remaining two terms in (26) are due to the presence of the external electromagnetic wave. In the absence of the wave the first term remains, while the last two terms drop out. These two terms are due to the mechanism of SESR. The second term is identified as the longitudinal SESR field and the third term as the transverse

SESR field. Obtaining the SESR contribution in two parts, namely longitudinal and transverse, is consistent with the conclusions drawn by Zachery [7]. The transverse SESR term matches with the result (equation (A28*b*)) derived in [5], except for a factor of γ^{-1} that comes into our expression because we have applied a relativistic correction. The longitudinal SESR term is additional here. It comes because we have explicitly taken in the calculations the electron's position changes due to the incident electromagnetic wave (instead of the mean position as is done in [5]). For the low frequency of the incident electromagnetic wave (up to microwaves), both the longitudinal and transverse parts of SESR are comparable, and for higher frequencies transverse SESR dominates over the longitudinal; but in no case is transverse SESR negligible as concluded in [7].

Appendix 1

Neglecting the magnetic field term in equation (4), we obtain the equation of motion of the electron as

$$\gamma m_0 \, \mathrm{d}\bar{V}/\mathrm{d}t = -e\bar{E}_0 \sin(\omega_0 t - \bar{K}_0 \cdot \bar{R}). \tag{A1.1}$$

To solve the above equation, we go to the Lorentz frame that moves with the velocity $\bar{u}$ (parallel to $\hat{z}$) with respect to the laboratory frame. In that frame we have denoted the electron's velocity by $\bar{v}'$, time by t', frequency by ω_0', and electric and magnetic field strengths, respectively, by $\bar{E}_0'$ and $\bar{B}_0'$ and so on. Then (A1.1) takes the following form:

$$\gamma m_0 \frac{\mathrm{d}\bar{v}'}{\mathrm{d}t} = -e\bar{E}_0' \sin(\omega_0' t' - \bar{K}_0' \cdot \bar{R}') - \frac{e}{c}(\bar{v}' \times \bar{B}_0') \sin(\omega_0' t' - \bar{K}_0' \cdot \bar{R}'). \tag{A1.2}$$

Here, even though $B_0' \sim \beta\gamma E_0'$ since $v' \ll c$, we can neglect the second term on the right-hand side of (A1.2) as compared to the first one. Integrating (A1.2), we obtain

$$\bar{v}'(t') = \frac{e\bar{E}_0'}{\gamma m_0} \frac{\cos(\omega_0' t' - \bar{K}_0' \cdot \bar{R}')}{\omega_0'}. \tag{A1.3}$$

Now, going back to the laboratory frame, we obtain

$$\bar{V}(t) = (0, \bar{v}'(t)/\gamma, u) \tag{A1.4}$$

where

$$\bar{v}'(t) = \frac{e(\gamma E_0)}{\gamma m_0} \frac{\cos(\omega_0 t - \bar{K}_0 \cdot \bar{R})}{[\gamma \omega_0 (1 + \beta n)]}.$$

Substituting $\bar{K}_0 = -(\omega_0 n/c)\hat{z}$, writing $\Omega = \omega_0(1+\beta n)$, $ut = \bar{R}\hat{z}$, $v_u = eE_0/m_0\Omega$ and $\beta = |\bar{V}/c| \sim |\bar{u}/c|$, we obtain the electron's velocity as

$$\bar{V}(t) = (0, (v_u/\gamma^2) \cos \Omega t, u). \tag{A1.5}$$

Integrating (A1.5) WRT t we obtain the electron's position as

$$\bar{R}(t) = (0, (v_u/\gamma^2 \Omega) \sin \Omega t, ut). \tag{A1.6}$$

Appendix 2

Let us denote the integral with respect to ϕ by I_ϕ

$$\int_0^{2\pi} \exp[iK_\rho\rho\cos(\phi-\phi')]J_l(K_\rho y_0\sin\phi)\,d\phi \equiv I_\phi. \tag{A2.1}$$

Expanding $\cos(\phi-\phi')$ and splitting the interval of I_ϕ, namely 0 to 2π, into two parts, namely 0 to π and π to 2π, we obtain

$$\begin{aligned} I_\phi = \int_0^{\pi} d\phi\, J_l(K_\rho y_0\sin\phi)\Bigg[\left(\frac{1+(-1)^l}{2}\right) & \\ \times\{\exp[i(a\cos\phi+b\sin\phi)]+\exp[-i(a\cos\phi+b\sin\phi)]\} & \\ +\left(\frac{1+(-1)^{l+1}}{2}\right)\{\exp[i(a\cos\phi+b\sin\phi)] & \\ -\exp[-i(a\cos\phi+b\sin\phi)]\}\Bigg] & \end{aligned} \tag{A2.2}$$

where $a=K_\rho\rho\cos\phi'$, $b=K_\rho\rho\sin\phi'$.

Reducing the interval of the integral in (A2.2) further to 0 to $\pi/2$ with some simplifications, we obtain

$$\begin{aligned} I_\phi = 2\int_0^{\pi/2} d\phi\Bigg[\left(\frac{1+(-1)^l}{2}\right)[J_l(K_\rho y_0\sin\phi)\cos(a\cos\phi+b\sin\phi) & \\ +J_l(K_\rho y_0\cos\phi)\cos(-a\sin\phi+b\cos\phi)] & \\ +i\left(\frac{1+(-1)^{l+1}}{2}\right)[J_l(K_\rho y_0\sin\phi)\sin(a\cos\phi+b\sin\phi) & \\ +J_l(K_\rho y_0\cos\phi)\sin(-a\sin\phi+b\cos\phi)]\Bigg\}. & \end{aligned} \tag{A2.3}$$

Using $a\cos\phi+b\sin\phi=K_\rho\rho\cos(\phi-\phi')$ and $-a\sin\phi+b\cos\phi=-K_\rho\rho\sin(\phi-\phi')$ in (A2.3) and simplifying, we obtain

$$\begin{aligned} I_\phi = 4\int_0^{\pi/2} d\phi\, J_0(K_\rho y_0\sin\phi)\cos[K_\rho\rho\cos(\phi-\phi')] & \\ +8\int_0^{\pi/2} d\phi\, J_l(K_\rho y_0\sin\phi)\cos[K_\rho\rho\cos(\phi-\phi')] & \end{aligned} \tag{A2.4}$$

where $l=+2, +4, \ldots.$

Putting $\phi'=0$, and using the standard integral given earlier, we obtain

$$I_\phi = 2\pi J_0(EK_\rho)J_0(FK_\rho)+4\pi J_{l/2}(EK_\rho)J_{l/2}(FK_\rho) \tag{A2.5}$$

where

$$E=\tfrac{1}{2}[(y_0^2+\rho^2)^{1/2}+\rho] \qquad F=\tfrac{1}{2}[(y_0^2+\rho^2)^{1/2}-\rho].$$

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